(#) Izračunati $I=\int_0^2 x\sqrt{4+x^2}\,\mathrm{arctg}\,\frac{x}{2}\,dx$.

Rj. $\int_0^2 x\sqrt{4+x^2}\,\mathrm{arc\,tg}\,\frac{x}{2}\,dx = \left| \begin{matrix} u=\mathrm{arctg}\,\frac{x}{2} & dv=x\sqrt{4+x^2} \\ du=\frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2} & v\overset{(*)}{=}\frac{1}{3}(4+x^2)^{\frac{3}{2}} \end{matrix} \right| \overset{(\square)}{=}$

$\int x\sqrt{4+x^2}\,dx = \left| \begin{matrix} 4+x^2=s^2 \\ 2x\,dx=2s\,ds \\ x\,dx=s\,ds \end{matrix} \right| = \int s^2\,ds = \frac{1}{3}s^3+C=\frac{1}{3}(4+x^2)^{\frac{3}{2}}+C \quad \ldots(*)$

$\overset{(\square)}{=} \frac{1}{3}(4+x^2)^{\frac{3}{2}}\,\mathrm{arc\,tg}\,\frac{x}{2}\Big|_0^2 - \frac{1}{3}\cdot\frac{1}{2}\int_0^2 \frac{1\;{\scriptstyle 1\cdot 4}}{1+\frac{x^2}{4}\;{\scriptstyle 1\cdot 4}}(4+x^2)^{\frac{3}{2}}\,dx =$

$= \frac{1}{3}\left[\sqrt{8^3}\,\mathrm{arctg}\,1 - 0\right] - \frac{1}{6}\cdot 4\int_0^2 \frac{(4+x^2)^{\frac{3}{2}}}{4+x^2}\,dx =$

$= \frac{1}{3}\cdot 8\cdot 2\sqrt{2}\cdot\frac{\pi}{4} - \frac{2}{3}\int_0^2 \sqrt{4+x^2}\,dx = \frac{4}{3}\pi\sqrt{2} - \frac{2}{3}\int_0^2\sqrt{4+x^2}\,dx \quad \ldots(1)$

Trebamo još izračunati $\int_0^2 \sqrt{4+x^2}\,dx$:

$J=\int_0^2\sqrt{4+x^2}\,dx = \left| \begin{matrix} u=\sqrt{4+x^2} & dv=dx \\ du=\frac{x}{\sqrt{4+x^2}}\,dx & v=x \end{matrix} \right| = x\sqrt{4+x^2}\Big|_0^2 - \int_0^2 \frac{x^2+4-4}{\sqrt{4+x^2}}\,dx$

$= 2\sqrt{8} - \int_0^2\sqrt{4+x^2}\,dx + 4\int_0^2\frac{dx}{\sqrt{x^2+4}} = 4\sqrt{2} - J + 4\ln|x+\sqrt{x^2+4}|\Big|_0^2 =$

$= 4\sqrt{2} - J + 4\left(\ln(2+2\sqrt{2}) - \ln 2\right) = 4\sqrt{2} - J + 4\ln(1+\sqrt{2}) \Rightarrow$

$2J = 4\sqrt{2} + 4\ln(1+\sqrt{2}) \Rightarrow J=\int_0^2\sqrt{4+x^2}\,dx = 2\sqrt{2}+2\ln(1+\sqrt{2}). \quad \ldots(2)$

$I=\int_0^2 x\sqrt{4+x^2}\,\mathrm{arctg}\,\frac{x}{2}\,dx \overset{(1);(2)}{=} \frac{4}{3}\pi\sqrt{2} - \frac{4}{3}\sqrt{2} - \frac{4}{3}\ln(1+\sqrt{2})$ traženo

$= \frac{4}{3}\left[\sqrt{2}(\pi-1) - \ln(1+\sqrt{2})\right]$ rješenje

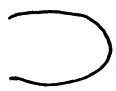
\# Izračunati zapreminu tijela koje nastaje rotacijom figure određene parabolom $y^2 = 9-3x$, tangentom na tu parabolu u tački $A(0,3)$ i x-osom oko x-ose.

Rj.

$y^2 = 9-3x$

$3x = -y^2+9$

$x = -\frac{1}{3}y^2+3$



f-ja je ovog oblika

$x' = -\frac{2}{3}y$

$x' = 0$ akko $y = 0$

$T(3,0)$ je tjeme f-je

$x = 0 \Rightarrow y = \pm 3$

Koeficijent pravca tangente

$k = x'(A) = -\frac{2}{3}\cdot 3 = -2$

$x - x_1 = k(y-y_1)$

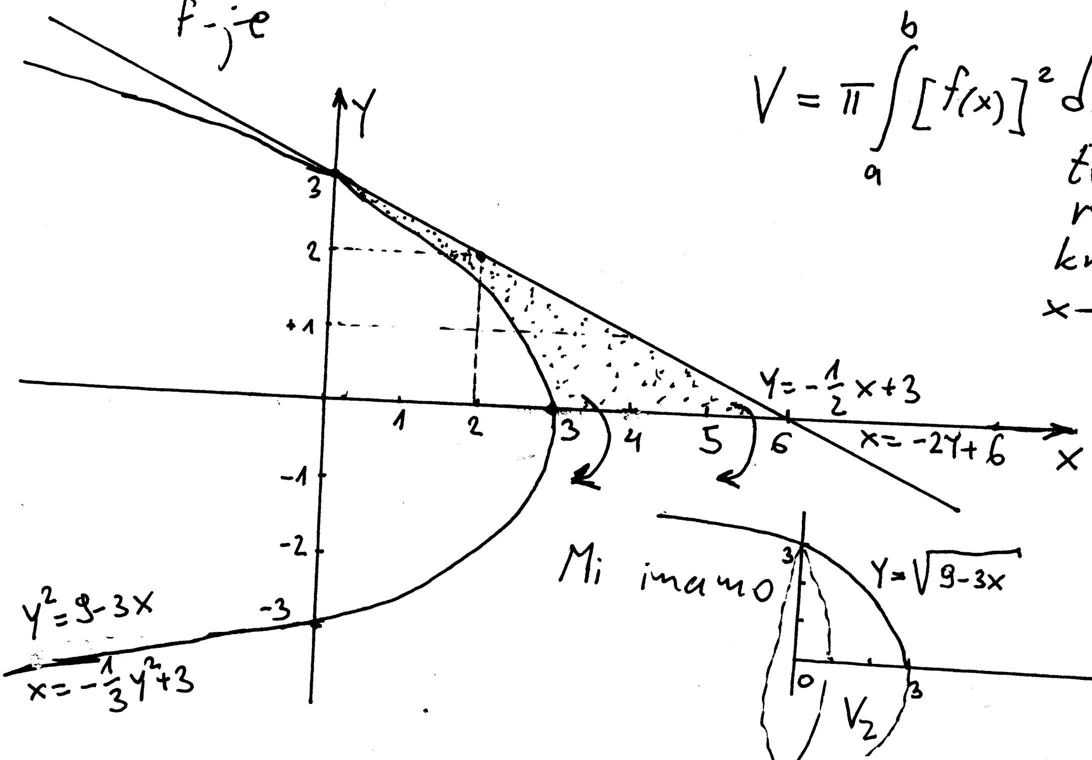
$x - 0 = (-2)(y-3)$

$x = -2y+6$

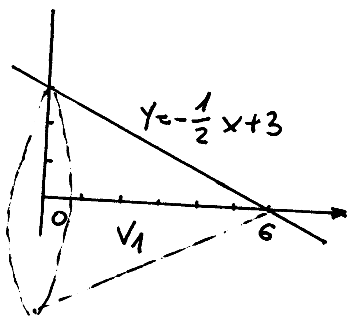
$-2y = x-6$

$y = -\frac{1}{2}x+3$

$V = \pi\int_a^b [f(x)]^2 dx$ je zapremina tijela dobijena rotacijom dijela krive $y = f(x)$ oko x-ose



Mi imamo



Zapremina našeg tijela se računa po formuli

$$V = V_1 - V_2 = \pi\int_0^6 \left(-\frac{1}{2}x+3\right)^2 dx - \pi\int_0^3 \left(\sqrt{9-3x}\right)^2 dx \overset{(1)\,i\,(2)}{=} \frac{9}{2}\pi$$

$$V_1 = \pi\int_0^6 \left(\frac{1}{4}x^2 - 3x + 9\right) dx = \pi\left(\frac{1}{4}\cdot\frac{1}{3}x^3\Big|_0^6 - 3\cdot\frac{1}{2}x^2\Big|_0^6 + 9x\Big|_0^6\right) = \pi(18 - 54 + 54) = 18\pi \quad \dots(1)$$

$$V_2 = \pi\int_0^3 (9-3x)dx = \pi\left(9x\Big|_0^3 - 3\cdot\frac{1}{2}x^2\Big|_0^3\right) = \pi\left(27 - \frac{27}{2}\right) = \frac{27}{2}\pi \quad \dots(2)$$

(#) Naći stacionarne tačke za uslovne ekstreme f-je $z = 2x^2 + 12xy + y^2$, ako je $x^2 + 4y^2 = 25$.

Rj. Formirajmo f-ju $F(x,y,\lambda) = 2x^2 + 12xy + y^2 + \lambda(x^2 + 4y^2 - 25)$.

$$\frac{\partial F}{\partial x} = 4x + 12y + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 12x + 2y + 8\lambda y$$

$$\frac{\partial F}{\partial \lambda} = x^2 + 4y^2 - 25$$

$$4x + 12y + 2\lambda x = 0 \quad /:2$$
$$12x + 2y + 8\lambda y = 0 \quad /:2$$
$$x^2 + 4y^2 - 25 = 0$$

$$(2+\lambda)x + 6y = 0 \quad \dots(1)$$
$$6x + (1+4\lambda)y = 0 \quad \dots(2)$$
$$x^2 + 4y^2 = 25 \quad \dots(3)$$

(1): $6y = -(2+\lambda)x$

$$\boxed{y = -\frac{1}{3}x - \frac{\lambda}{6}x}$$

$$y = -\frac{1}{6}(2+\lambda)x.$$

(1) uvrstimo u (2)

$$6x + (1+4\lambda)y = 0$$
$$6x + (1+4\lambda)\left(-\frac{1}{6}\right)(2+\lambda)x = 0 \quad /\cdot(-6)$$
$$-36x + (1+4\lambda)(2+\lambda)x = 0$$
$$(-36 + (1+4\lambda)(2+\lambda))x = 0$$
$$(-36 + 2 + 9\lambda + 4\lambda^2)x = 0$$
$$(4\lambda^2 + 9\lambda - 34)x = 0$$
$$x = 0 \quad \text{ili} \quad 4\lambda^2 + 9\lambda - 34 = 0$$

Ako bi x bilo jednako nuli tada bi na osnovu (1) imali da je $y=0$ a kako mora biti $x^2 + 4y^2 = 25$ to x ne može biti nula.

Prema tome mora biti $4\lambda^2 + 9\lambda - 34 = 0$.

$$4\left(\lambda + \frac{17}{4}\right)(\lambda - 2) = 0$$

$D = 81 + 544 = 625$

$$\lambda_{1,2} = \frac{-9 \pm 25}{8} \qquad \lambda_1 = \frac{-34}{8} = -\frac{17}{4} \qquad \lambda_2 = \frac{16}{8} = 2$$

Za $\lambda = 2$ na osnovu (1) $y = -\frac{1}{6}\cdot 4x = -\frac{2}{3}x$. Ako ovaj rezultat uvrstimo u (3) dobićemo $x^2 + 4\cdot\frac{4}{9}x^2 = 25 \Rightarrow \frac{25}{9}x^2 = 25$

$x^2 = 9 \quad x_{1,2} = \pm 3$

Za $\lambda = 2$ imamo dvije stacionarne tačke $M_1(-3,2)$ i $M_2(3,-2)$.

Za $\lambda = -\frac{17}{4}$ na osnovu (1) $y = \left(-\frac{1}{6}\right)\left(2 - \frac{17}{4}\right)x = \left(-\frac{1}{\cancel{6}_2}\right)\left(-\frac{\cancel{9}^3}{4}\right)x = \frac{3}{8}x$. Ako ovaj rezultat uvrstimo u (3) dobićemo $x^2 + 4\cdot\frac{9}{64}x^2 = 25$ tj. $x^2 + \frac{9}{16}x^2 = 25$

$\Rightarrow \frac{25}{16}x^2 = 25 \Rightarrow x^2 = 16 \Rightarrow x_{3,4} = \pm 4.$

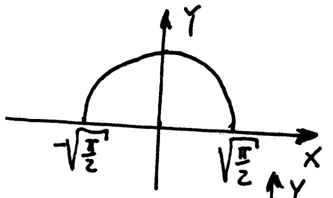
Za $\lambda = -\frac{17}{4}$ imamo dvije stacionarne tačke $M_3\left(-4, -\frac{3}{2}\right)$ i $M_4\left(4, \frac{3}{2}\right)$.

\# Izračunati dvostruki integral

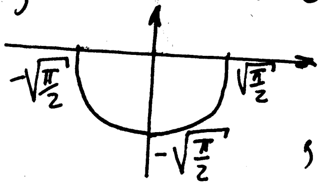
$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy\,.$$

Rj. Oblast integracije D je $D:\begin{cases}0\le x\le\sqrt{\frac{\pi}{2}}\\ -\sqrt{\frac{\pi}{2}-x^2}\le y\le\sqrt{\frac{\pi}{2}-x^2}\end{cases}$

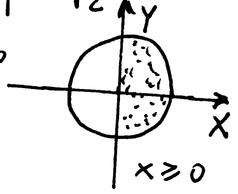
Znamo da f-ja $y=\sqrt{\frac{\pi}{2}-x^2}$ izgleda ovako



dok f-ja $y=-\sqrt{\frac{\pi}{2}-x^2}$ izgleda



, sve zajedno $\Rightarrow$



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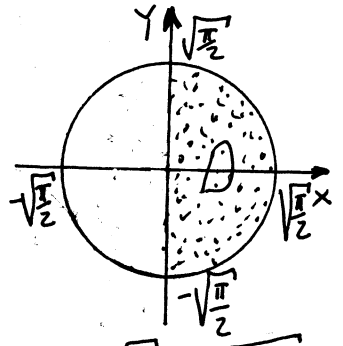
Ove dvije f-je se dobiju iz f-je $y^2=\frac{\pi}{2}-x^2$ tj. $x^2+y^2=\frac{\pi}{2}$ što predstavlja jednačinu kruga sa centrom u koordinatnom početku, poluprečnika $\sqrt{\frac{\pi}{2}}$.



Uvedimo polarne koordinate

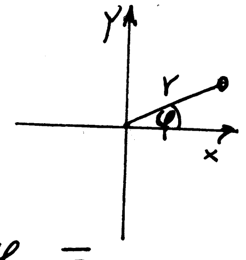
$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$x^2+y^2=\ldots=r^2$

$D\xrightarrow{\text{transform.}}D':\begin{cases}0\le r\le\sqrt{\frac{\pi}{2}}\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$



$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy=\iint_D\cos(x^2+y^2)\,dx\,dy=\iint_{D'}\cos(r^2)\,r\,dr\,d\varphi=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{\sqrt{\frac{\pi}{2}}}r\cos(r^2)\,dr=\left|\begin{array}{l}r^2=t\\ 2r\,dr=dt\\ r\,dr=\frac{1}{2}dt\\ r\Big|_0^{\sqrt{\frac{\pi}{2}}}\Rightarrow t\Big|_0^{\frac{\pi}{2}}\end{array}\right|=\frac{1}{2}\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi/2}\cos t\,dt=\frac{1}{2}\cdot\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\sin t\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\cdot\pi\cdot 1=\frac{\pi}{2}$$ traženo rješenje